

First Year Algebra Exam – May 2007

Time allowed: 240 minutes

Do **seven** of the following ten problems. Please do not turn in more than seven problems.

You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, \mathbf{Z} , resp. \mathbf{N} , \mathbf{Q} , \mathbf{C} , is the set of all integers, resp. positive integers, rational numbers, complex numbers.

1. State the three Sylow theorems and prove the existence of Sylow subgroups.
2. Let H be a normal subgroup of G . Assume H is cyclic. Prove that any subgroup of H is also normal in G .
3. Classify, up to isomorphism, the groups of order 28 with non-cyclic Sylow 2-subgroups.
4. Prove that any nonzero vector space over a field \mathbf{F} has a basis. (You may assume Zorn's Lemma).
5. Let $R = \mathbf{Z}[x]$ be the ring of polynomials in variable x with integer coefficients. Let $I = (5, x^2 + 2)$ be the ideal of R generated by 5 and $x^2 + 2$. Prove that R/I is a finite field, and find its cardinality.
6. Consider the ring $\mathbf{F}[x, y]$ of polynomials in two commuting variables x, y over a field \mathbf{F} . Is this ring (i) a Euclidean domain? (ii) a principal ideal domain? (iii) a unique factorization domain? Justify your answer(s).
7. Let A be an $n \times n$ -matrix with entries in \mathbf{Q} . Suppose A is invertible and $A^4 - 4A = 2A^{-1}$. Show that n is divisible by 5 and that such a matrix A is unique up to similarity.
8. Let T and S be linear transformations $\mathbf{C}^5 \rightarrow \mathbf{C}^5$, both with the same characteristic polynomial $x^2(x^2 + 1)(x - 3)$. Assume that their kernels, $\text{Ker}(T)$ and $\text{Ker}(S)$, have the same dimension. Are T and S necessarily similar? Justify your answer.
9. Let $p \in \mathbf{N}$ be a prime. Find a splitting field \mathbf{K} for $x^p - p$ over \mathbf{Q} , and determine $[\mathbf{K} : \mathbf{Q}]$.
10. Let \mathbf{F} be a subfield of a field \mathbf{K} and let $\alpha, \beta \in \mathbf{K}$ be algebraic over \mathbf{F} , of degree m and n , respectively. Assume m and n are coprime. Find the degree $[\mathbf{F}(\alpha, \beta) : \mathbf{F}]$.