First Year Exam in Algebra (January 2007)

Time allowed: Four Hours.

Answer **seven** problems. You should indicate which problems you wish to be graded. **Write your answers clearly in complete English sentences.** You may quote results (within reason) as long as you state them clearly.

1. Construct two nonisomorphic nonabelian groups of order $2007 = 3^2.223$.

2. Let $p \in \mathbb{Z}$ be a prime.
(a) Show that if $p \equiv 1 \bmod 4$, then p divides an integer of the form $n^2 + 1$. (Hint: Consider $(\mathbb{Z}/p\mathbb{Z})^\times$.)
(b) Show, using (a), that if $p \equiv 1 \bmod 4$, then p is not irreducible in the ring $\mathbb{Z}[i]$ of Gaussian integers.
(c) Show that if $p \equiv 3 \bmod 4$ then p is irreducible in $\mathbb{Z}[i]$.

3. Determine the conjugacy classes of the dihedral group of order 28.

4.(a) Show that left multiplication defines an action of a group G on itself and explain how this action gives rise to a homomorphism from G to the group S_G of permutations of the set G.
(b) Let G be a finite group of even order whose Sylow 2-subgroups of G are cyclic. Show that G has a normal subgroup of index 2. Hint: Find an element which gives an odd permutation in (a).

5. Let F be a field and $p(x) \in F[x]$ be an irreducible polynomial of degree n.
(a) Prove that there exists a field extension E of F of degree n containing a root of $p(x)$.
(b) Define the term *splitting field* of a polynomial in $F[x]$. Prove, using (a) that any polynomial of degree n in $F[x]$ has a splitting field of degree $\leq n!$ over F.

6. Let R be a ring with identity and let M be an R-module. Suppose A, B and C are R-submodules of M.
(a) Prove that $A + B$ and $A \cap B$ are submodules of M.
(b) Prove that if $C \subseteq A$, then $A \cap (B + C) = A \cap B + C$.
(c) Give an example to show that the hypothesis $C \subseteq A$ of (b) is necessary.

7.(a) State Zorn's Lemma.
(b) Prove that every vector space has a basis.

8. Define the ring $F[[x]]$ of formal power series over a field F and show that it is a principal ideal domain.

9. Let G be a finite group of order p^a, where p is a prime and $a > 1$.
(a) Show that the center of G is not trivial.
(b) Deduce that G has normal subgroups of order p^b for all $b = 0, \dots, a$.

10. Let V be a finite-dimensional vector space over a field F.
(a) What do we mean by the *minimal polynomial* of a linear transformation T of V.
(b) Prove that the minimal polynomial of T can be factorized into distinct linear factors if and only if V has a basis in which T is represented by a diagonal matrix.