MASTER'S EXAM IN ALGEBRA

Time allowed: Four Hours.

Answer **seven** problems. You should indicate which problems you wish to be graded. **Write your answers clearly in complete English sentences.** You may quote results (within reason) as long as you state them clearly.

1. (a) State the Chinese Remainder Theorem for commutative rings with 1.
(b) Using (a), or otherwise, show that if m and n are relatively prime positive integers, then

$$(\mathbb{Z}/mn\mathbb{Z})^\times \cong (\mathbb{Z}/m\mathbb{Z})^\times \times (\mathbb{Z}/n\mathbb{Z})^\times,$$

(As usual, $(\mathbb{Z}/d\mathbb{Z})^\times$ denotes the group of units in $\mathbb{Z}/d\mathbb{Z}$.)
(c) Let ϕ be the Euler function, i.e. $\phi(n) =$ the number of positive integers less than n and relatively prime to n. Show that if m and n are relatively prime, then $\phi(mn) = \phi(m)\phi(n)$.

2. Let R be a ring with 1 and F a free (left) R-module, with basis X.
(a) State the universal mapping property of F.
(b) Suppose $M \xrightarrow{\phi} F$ is a surjective homomorphism of (left) R-modules. Show that there exists an R-module homomorphism $F \xrightarrow{\sigma} M$ such that $\phi \circ \sigma = \mathrm{id}_F$.
(c) Show that in (b), M is the direct sum of the kernel of ϕ and the image of σ.

3. Let G be a group of order 56.
(a) Prove that G has either a normal Sylow 2-subgroup or a normal Sylow 7-subgroup.
(b) Construct either a group of order 56 with no normal Sylow 2-subgroup or a group of order 56 with no normal Sylow 7- subgroup.

4. Classify the conjugacy classes in the group $\mathrm{GL}(4, \mathbb{F}_2)$ of invertible 4×4 matrices over the field of two elements.

5. Let K be a field extension of F. Let

$$A = \{a \in K : a \text{ is algebraic over } F\}.$$

Prove that A is a subfield of K.

6. Let $R = \mathbb{Z}[\sqrt{-7}]$. Prove that R is not a unique factorization domain.

7. Let K be a splitting field over $\mathbb{Q}$ of the polynomial $x^4 + 2 \in \mathbb{Q}[x]$. Find $|K : \mathbb{Q}|$. (Hint: You may assume that K is a subfield of the field of complex numbers.)

8. (a) Define the terms *characteristic subgroup* and *inner automorphism* of a group.
(b) Let G be a finite group, let P be a Sylow p-subgroup and let $N = N_G(P)$. Show that $N_G(N) = N$.

9. Prove that the ring $\mathbb{Z}[i]$ of gaussian integers is a unique factorization domain. (You may quote any general theorems which you need, as long as you state them clearly and correctly.)

10. Let V be a vector space of finite dimension n over a field F. A linear endomorphism T of V is *nilpotent* if $T^N = 0$ for some natural number N.

By considering either the Jordan canonical form or the rational canonical form, show that the similarity classes of nilpotent endomorphisms of V are in one-one correspondence with partitions of n.

11. Let V be a vector space over a field F and W a subspace. Show that there is a subspace U of V such that V is the direct sum of W and U. (Do not assume V is finite dimensional. If you want to use the fact that every vector space has a basis, or any similar result, you need to prove this result in detail.)