Algebra First Year Examination
January, 2006

Instructions. Work any seven of the following 11 problems. Please do not turn in more than seven. Please start each problem on a new page and write on one side of the page. Thank you.

1. (a) Compute the order of the general linear group $\mathrm{GL}_n(\mathbb{Z}_p)$, with p a prime number.
   (b) Calculate the order of the subgroup $\mathrm{SL}_n(\mathbb{Z}_p)$ of matrices which have determinant 1.

2. Suppose that $\phi : G \longrightarrow H$ is a surjective homomorphism of groups. Prove the following about the assignment $\phi^* : N \mapsto \phi^{-1}(N)$. Assume that it maps subgroups to subgroups.
   (a) ϕ^* is a bijection between the lattice of subgroups of H and the set of subgroups of G that contain $Ker(\phi)$.
   (b) $N_1 \subseteq N_2$ if and only if $\phi^*(N_1) \subseteq \phi^*(N_2)$.
   (c) $\phi^*(N)$ is normal in G if and only if N is normal in H.

3. Let G be a group of order 110. Prove that there is a unique Sylow 11-subgroup and a normal subgroup of order 55. But also give an example showing that the Sylow 5-subgroups need not be normal.

4. Prove that A_5 is a simple group.

5. Consider $A = \mathbb{R}^{\mathbb{N}}$, the ring of all real valued sequences, under pointwise operations. Prove:
   (a) for each $n \in \mathbb{N}$, $M_n = \{f \in A : f(n) = 0\}$ is a maximal ideal of A;
   (b) there exist maximal ideals besides the M_n $(n \in \mathbb{N})$. (Zorn's Lemma)

6. A ring R is *boolean* if $x^2 = x$, for each $x \in R$. Prove the following.
   (a) Every boolean ring has characteristic 2 and is commutative.
   (b) Assume R is a boolean ring with identity. Prove that every prime ideal is maximal.
   (c) Prove that every finite boolean ring with identity $1 \neq 0$ has 2^n elements, for a suitable positive integer n.

7. Give examples of the following, and justify your choices:
   (a) A unique factorization domain which is not a principal ideal domain.
   (b) A local integral domain with a *nonzero* prime ideal that is not maximal.
   (c) An integral domain in which the uniqueness provision of "unique factorization" fails.

8. Let R be a ring with identity. If F is a free R-module of rank $n < \infty$, then show that $\mathrm{Hom}_R(F, M)$ is isomorphic to M^n, as an R-module, for each R-module M.

9. Suppose that $T : V \longrightarrow V$ is a linear transformation on the vector space V. Call T a *projection* if $T^2 = T$. Prove that if T is a projection then $V = \ker(T) \oplus T(V)$.

   Give an example to show that the converse of the above proposition is false.

10. Over $\mathbb{Q}$, consider the following matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Answer the following:

(a) Prove that A is diagonalizable.

(b) Is the $\mathbb{Q}[x]$-module structure on $\mathbb{Q}^4$ by A, via

$$f(x)v = f(A)v, \quad \text{for each} \quad v \in \mathbb{Q}^4,$$

cyclic? Explain.

11. Suppose that F is a subfield of K and K a subfield of L, so that the dimensions $[K : F]$ and $[L : K]$ are finite. Prove that

$$[L : F] = [L : K][K : F].$$

Now suppose that $[F(u) : F]$ is odd; prove that $F(u) = F(u^2)$.