Algebra First Year Examination; September, 2005

Answer seven questions; please do not turn in more than seven. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. State and prove the First Isomorphism Theorem for groups.
2. Suppose that G is a group and H is a proper subgroup of index $k > 1$.
   (a) Show that $g * (xH) = gxH$ defines a group action of G on the set $\Omega = G/H$ of left cosets of H.
   (b) Prove that the kernel of the induced homomorphism of G into the permutation group on Ω is the intersection of all the conjugates of H.
   (c) Now suppose that G is simple. Show that G is isomorphic to a subgroup of S_k.
3. Classify the groups of order 70.
4. Describe the Division Algorithm for the ring $\mathbb{Z}[i]$ of Gaussian integers. Prove that $\mathbb{Z}[i]$ is a Euclidean ring.
5. Using Zorn's Lemma, prove that in each commutative ring with $1 \neq 0$ every prime ideal contains a minimal prime ideal.
6. Let G be a group and let $K \leq H \leq G$.
   (a) Define what it means for H to be a characteristic subgroup of G.
   (b) Prove that if H is a normal subgroup of G and K is a characteristic subgroup of H then K is a normal subgroup of G.
   (c) Give an example which shows that, in the situation described in (b), K need not be a characteristic subgroup of G.
   (d) Give an example which shows that if H is a characteristic subgroup of G and K is a normal subgroup of H, then K need not be a normal subgroup of G.
7. Consider the polynomial X^2+1 over the field $\mathbb{Z}_7$. Prove that $E = \mathbb{Z}_7[X]/(X^2+1)$ is a field of 49 elements.
8. Suppose that F is a field and G is a finite multiplicative subgroup of $F \setminus \{0\}$. Prove that G is cyclic.
9. Prove that $\mathbb{Q}$, the additive group of the rationals, is not a free abelian group. (Hint: It might be useful to first establish that $\mathbb{Q}$ is not cyclic.)

10. Let F be a field, let V be a finite-dimensional vector space over F, and let $T : V \to T$ be a linear transformation.

(a) Suppose that the characteristic polynomial of T splits over F. Show how to determine the minimum polynomial of T from the Jordan Canonical Form of T. Prove your assertions.

(b) Prove that T is diagonalizable if and only if $m_T(X)$, the minimum polynomial of T, can be factored as

$$m_T(X) = (X - \lambda_1)(X - \lambda_2) \cdots (X - \lambda_k),$$

where the $\lambda_i \in F$ $(i = 1, \ldots, k)$ are distinct.

11. Let V be a finite dimensional vector space over $\mathbb{Q}$ and let T be an invertible linear transformation on V.

(a) Prove that if $T^{-1} = T^2 + T$ then the dimension of V is a multiple of 3.

(b) Prove that if $\dim(V) = 3$ then all transformations T on V such that $T^{-1} = T^2 + T$ are similar.