

Algebra First Year Examination; May 17th, 2005

Answer seven questions; please do not turn in more than seven. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Prove that if G is a nontrivial finite p -group, then its center is not trivial. Use this fact to prove that every finite p -group is nilpotent.
2. Prove that any subgroup of a cyclic group is cyclic.
3. (a) Show that if G is a subgroup of S_n (n a natural number) containing an odd permutation, then half the elements of G are odd and half are even.
(b) Use (a) to prove that if G is a group of order $2m$, with $m \geq 3$ odd, then G cannot be simple, and, indeed, contains a subgroup of index 2. (Hint: Let G act on itself by left multiplication.)
4. Let G be a group of $385 = 5 \cdot 7 \cdot 11$ elements. Prove that the Sylow 11-subgroups are normal, and that any Sylow 7-subgroup of G lies in the center. Also give an example to show that the Sylow 11-subgroup need not be in the center.
5. Using Zorn's Lemma, prove that each commutative ring with $1 \neq 0$ has at least one maximal ideal.
6. Let F be a field and $F[[X]]$ denote the ring of all formal power series in the indeterminate X with coefficients in F . Show that $F[[X]]$ is a Euclidean domain. (This asks you to define a norm on $F[[X]]$, and then to show that it satisfies the 'division algorithm' provision.)
7. Let n be a natural number; prove that the polynomial

$$\Phi_n(X) = \frac{X^n - 1}{X - 1}$$

is irreducible over the ring \mathbb{Z} of integers precisely when n is prime.

8. Let R be a ring with identity. Call a left R -module *Noetherian* if it satisfies the ascending chain condition on submodules. Prove that if M is an R -module having a Noetherian R -submodule N , such that M/N is also Noetherian, then M is Noetherian.
9. Let R be a ring with identity. Suppose that $\phi : M \rightarrow F$ is a surjective R -homomorphism and that F is a free R -module. Prove that there is a submodule N of M such that $N \cong F$ and $M = \ker(\phi) \oplus N$.
10. Prove, over any field F , that if two 2×2 matrices or two 3×3 matrices have the same minimum and characteristic polynomials then they are similar matrices.
Give an example which shows that this is false for 4×4 matrices.
11. Prove that if F is a finite field, then there is a prime number p and a natural number n , so that F has p^n elements.