First Year Algebra Exam **September 9, 2004, 6:00 p.m. - 10:00 p.m.**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let G be a finite group and let H be a solvable subgroup of G such that $[G : H] \leq 4$. Prove that G is solvable.

2. How many isomorphism classes of groups of order 28 are there? Justify your answer.

3. Prove that if p is a prime and P is a group of order p^n, for some $n \geq 1$, then the center of P is not trivial. Deduce that P is nilpotent.

4. Prove that group $\text{Aut}(\mathbf{Z}/k\mathbf{Z})$ is isomorphic to $U(k)$ of all integers i modulo k such that $0 \leq i < k$ and i is relatively prime to k, under multiplication modulo k.

5. Let R be a non-zero commutative ring with identity. Define what it means for an ideal I of R to be a *minimal prime* ideal of R. Use your definition to prove that R has at least one minimal prime ideal.

6. Let R be a commutative ring with identity. If F is a free R-module of rank $n < \infty$, then show that $\text{Hom}_R(F, M) \cong M^n$, for each R-module M.

7. Let K be a field extension of F. Let
$$A = \{a \in K : a \text{ is algebraic over } F\}.$$
Prove that A is a field extension of F.

8. Let $R = \mathbf{Z}[\sqrt{-7}]$. Prove that R is not a unique factorization domain.

9. Let $F = \mathbf{Z}/2\mathbf{Z}$ and let
$$f(x) = (x^2 + 1)(x^2 + x + 1) \in F[x].$$
Let K be the splitting field of $f(x)$ over F. Calculate $|K|$. Justify your answer carefully.

10. Let V be a finite dimensional real vector space. Let $T : V \to V$ be an invertible linear transformation such that $T^{-1} = -T^3 - 2T$. Prove that the dimension of V is even.