First Year Algebra Exam **May 13, 2004, 6:00 p.m. - 10:00 p.m.**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. How many isomorphism types of abelian groups of order $2^4 \cdot 3^3$ are there? Justify your answer.

2. Prove that every group of order 2004 is solvable. (Hint: the prime factorization of 2004 is $2004 = 2^2 \cdot 3 \cdot 167$.)

3. State the three Sylow theorems and prove the First Sylow Theorem about the existence of Sylow subgroups.

4. How many conjugacy classes of elements of order 10 does the symmetric group S_9 have? How many conjugacy classes of elements of order 10 does the alternating group A_9 have? Justify your answers.

5. Let R be a commutative ring with identity different from zero. Prove that R has a maximal ideal.

6. Let F be a field. Prove that a non-zero polynomial with coefficients in F and of degree n has at most n distinct roots in F.

7. Let M be a module over a commutative ring R. Define what is the *torsion part* of M. Prove that, if R is an integral domain, then the torsion part of M is a submodule of M. Give an example to show that the torsion part of M may fail to be a submodule when R is not an integral domain.

8. Let $R = \mathbf{Z}[x]$ be the ring of polynomials in one variable with integer coefficients. Let $I = (3, x^2 + x - 1)$ be the ideal of R generated by 3 and $x^2 + x - 1$. Prove that R/I is a finite field, and calculate the number of elements of R/I.

9. Find a splitting field K for $x^4 - 5$ over $\mathbf{Q}$, and determine $[K : \mathbf{Q}]$.

10. How many conjugacy classes of elements of order 5 are there in the group $GL(6, 2)$ of 6 by 6 invertible matrices over $\mathbf{Z}/2\mathbf{Z}$? Justify your answer and give a representative for each conjugacy class.