

## First Year Algebra Exam

January 22, 2004

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let  $n \geq 1$  and let  $p$  be prime. Determine the order of  $GL_3(\mathbb{Z}/p^n\mathbb{Z})$ , the unit group of the ring  $M_{3 \times 3}(\mathbb{Z}/p^n\mathbb{Z})$ . (Hint: Apply the first isomorphism theorem to the homomorphism  $\pi : GL_3(\mathbb{Z}/p^n\mathbb{Z}) \rightarrow GL_3(\mathbb{Z}/p\mathbb{Z})$ .)
2. Prove that there is no simple group of order  $6669 = 3^3 \cdot 13 \cdot 19$ .
3. Determine all groups of the form  $Z_8 \rtimes_{\phi} Z_2$  up to isomorphism. Be sure to prove that the groups on your list are pairwise non-isomorphic.
4. Let  $\sigma$  be a  $k$ -cycle in the symmetric group  $S_n$ . Give an explicit description of the centralizer of  $\langle \sigma \rangle$  in  $S_n$ . In particular, determine the order of the centralizer.
5. Let  $F$  be a field, let  $V$  be a vector space over  $F$ , and let  $S$  be a linearly independent subset of  $V$ . Use Zorn's Lemma to prove that there is a basis  $B$  for  $V$  which contains  $S$ . (Do not assume that every vector space has a basis.)
6. Let  $K$  be a field and let  $G$  be a finite subgroup of the multiplicative group of  $K$ . Prove that  $G$  is cyclic.
7. Let  $R$  be a unique factorization domain. Prove that  $R[X]$  is a unique factorization domain.
8. Let  $R$  be a ring with 1 and let  $M$  be a left  $R$ -module. Let
$$T(M) = \{x \in M : rx = 0 \text{ for some } r \in R \setminus \{0\}\}.$$
  - (a) Prove that if  $R$  is an integral domain then  $T(M)$  is a submodule of  $M$ .
  - (b) Give an example in which  $T(M)$  is not a submodule of  $M$ .
9. Let  $R$  be a ring with 1 and let  $S$  be a set.
  - (a) Give the definition of a free left  $R$ -module on  $S$ .
  - (b) Prove that there exists a free left  $R$ -module  $F(S)$  on  $S$ .
  - (c) Let  $M$  be a left  $R$ -module and let  $\phi : S \rightarrow M$  be a set map. Prove that there is a unique  $R$ -module homomorphism  $\tilde{\phi} : F(S) \rightarrow M$  such that  $\tilde{\phi}|_S = \phi$ .
10. Find a representative for each similarity class of  $6 \times 6$  matrices over  $\mathbb{C}$  with characteristic polynomial  $(X^4 - 1)(X^2 - 1)$ .