First Year Algebra Exam **September 4, 2003**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. (a) Define the commutator subgroup $G' = [G, G]$ of a group G.
   (b) Prove that G' is a characteristic subgroup of G.
   (c) Prove that if A is an abelian group and $\phi : G \to A$ is a homomorphism then there is a unique homomorphism $\overline{\phi} : G/G' \to A$ such that $\phi = \overline{\phi} \circ \pi$, where $\pi : G \to G/G'$ is the projection.

2. Let G be a group, let $N \trianglelefteq G$, and let $H \leq G$. Assume that G/N and H are finite, and that $\gcd(|G/N|, |H|) = 1$. Prove that $H \leq N$.

3. Prove that the alternating group A_n is simple for $n \geq 6$. You may assume that A_5 is simple.

4. Let G be a cyclic group of order n. Prove that $\mathrm{Aut}(G) \cong (\mathbb{Z}/n\mathbb{Z})^{\times}$.

5. Determine all isomorphism classes of groups of order $238 = 2 \cdot 7 \cdot 17$.

6. Prove that the ring of Gaussian integers $\mathbb{Z}[i]$ is Euclidean.

7. Let R be a commutative ring with $1 \neq 0$. Prove that the set of prime ideals of R contains a minimal element.

8. Let R be a UFD with field of fractions F and let $p(X) \in R[X]$. Prove Gauss' Lemma: If $p(X) = A(X)B(X)$ with $A(X), B(X) \in F[X]$ then there are $r, s \in F$ such that $a(X) = rA(X)$ and $b(X) = sB(X)$ both lie in $R[X]$ and $p(X) = a(X)b(X)$.

9. Let V be a vector space over a field F.
   (a) Define the dual space V^*.
   (b) Define the natural homomorphism $\phi : V \to V^{**}$.
   (c) Prove that if V is finite-dimensional then ϕ is an isomorphism.

10. Find a representative for each conjugacy class of elements of order 6 in the group $\mathrm{GL}_4(\mathbb{Q})$.