First Year Algebra Exam **May 22, 2003**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let G be a group of order $195 = 3 \cdot 5 \cdot 13$. Prove that G contains a normal subgroup of order 13 and a subgroup of order 5 which is contained in $Z(G)$. Give an example of a group of order 195 which is not abelian.

2. Let p be a prime of the form $p = 4k + 1$. Classify groups of order $4p$ up to isomorphism.

3. Let G be an abelian group which is simple. Prove that G is cyclic of prime order. (Do not assume that G is finite.)

4. Let G be a nontrivial finite p-group. Prove that $Z(G)$ is nontrivial.

5. Prove that the alternating group A_5 is simple.

6. Let R be a ring with 1 and let $I \neq R$ be an ideal in R. Prove that I is contained in a maximal ideal of R.

7. (a) Let R be a commutative ring with $1 \neq 0$ and let I, J be ideals in R. State the Chinese Remainder Theorem for R, I, J. Be sure to give the hypotheses that I and J must satisfy.

   (b) Find an integer n such that $n \equiv 2 \pmod{79}$ and $n \equiv -3 \pmod{164}$. (You don't need to carry out the arithmetic in your final answer.)

8. Let F be a field. Prove that the ring $F[X, Y]$ is not a PID by showing that some ideal $I \subset F[X, Y]$ is not principal.

9. Let R be a ring with $1 \neq 0$ and let M be a (unitary) left R-module. Prove that M is isomorphic to a quotient of a free module.

10. Find a representative for each similarity class of matrices $A \in \mathrm{M}_{4\times 4}(\mathbb{R})$ such that $A^4 = 0$.