

First Year Algebra Exam – Jan. 2003

Time allowed: 240 minutes

Do seven of the following ten problems. Please do not turn in more than seven problems. The problems do not have to be attempted in the order they are listed.

You must show your work. Answers with no work and/or no explanations will receive no credit. State clearly any theorem you use in your proofs.

In the problems, \mathbf{Z} , resp. \mathbf{Q} , \mathbf{C} , is the set of all integers, resp. rational numbers, complex numbers.

1. State the three Sylow theorems and prove the existence of Sylow subgroups.
2. Is there any simple group of order 105? Justify your answer.
3. Classify the groups of order $2 \cdot 2003$ up to isomorphism. (Hint : 2003 is a prime.)
4. Prove that if R is a ring with identity, then R has a maximal ideal. (You may assume Zorn's Lemma).
5. Consider the ring $S = \left\{ \begin{pmatrix} a & b \\ 5b & a \end{pmatrix} \mid a, b, c, d \in \mathbf{Z} \right\}$ (under matrix addition and matrix multiplication).
 - a) Let T be the subring $T = \{x + y\sqrt{5} \mid x, y \in \mathbf{Z}\}$ of the field \mathbf{R} of real numbers. Show that the rings S and T are isomorphic.
 - b) Using a) if you wish, show that S is an integral domain.
6. Give an example of a ring which is a unique factorization domain but not an Euclidean domain. Justify your example.
7. Let T and S be linear transformations $\mathbf{C}^5 \rightarrow \mathbf{C}^5$, both with the same characteristic polynomial $x^2(x-1)(x-2)(x-3)$. Assume that their ranges, $Im(T)$ and $Im(S)$, have the same dimension. Are T and S necessarily similar? Justify your answer.
8. How many conjugacy classes of elements of order 3 are there in the group $GL(4, \mathbf{Q})$ of invertible 4×4 -matrices over \mathbf{Q} ? Justify your answer and give a representative for each conjugacy class.
9. Find a splitting field \mathbf{K} for $x^5 - 5$ over \mathbf{Q} , and determine $[\mathbf{K} : \mathbf{Q}]$.
10. Let $\mathbf{F} = \mathbf{Q}(\sqrt[3]{2}, \sqrt[3]{3}, \dots, \sqrt[3]{2003})$ be the field obtained from \mathbf{Q} by joining all the numbers $\sqrt[3]{n}$, $2 \leq n \leq 2003$. Prove that $\sqrt[3]{7} \notin \mathbf{F}$.