First Year Algebra Exam – Sept. 2002

Time allowed: 240 minutes

Do **seven** of the following ten problems. Please do not turn in more than seven problems. The problems do **not** have to be attempted in the order they are listed.

You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, $\mathbf{Z}$, resp. $\mathbf{Q}$, $\mathbf{C}$, is the set of all integers, resp. rational numbers, complex numbers.

1. State and prove Cayley's Theorem about finite groups.

2. Let G be a group of order 44. Prove that
(i) a Sylow 11-subgroup of G is normal in G, and
(ii) the center of G contains an element of order 2.

3. Classify the groups of order $57 = 19 \cdot 3$ up to isomorphism.

4. Prove the Chinese Remainder Theorem: *Let R be a commutative ring with identity. Let I, J be ideals of R such that $I + J = R$. Then the homomorphism $\varphi : R \to R/I \times R/J$, $\varphi(a) = (a + I, a + J)$, is surjective, and its kernel is $I \cap J$.*

5. Let $\mathbf{F}[x, y]$ be the ring of (commutative) polynomials in two variables x, y over a field $\mathbf{F}$. Is $\mathbf{F}[x, y]$ (i) A Euclidean domain? (ii) A principal ideal domain? (iii) A unique factorization domain? Justify your answer.

6. Using Eisenstein's criterion, prove that there are infinitely many irreducible monic polynomials of degree 2002 over $\mathbf{Q}$. Is the statement still true if one replaces $\mathbf{Q}$ by $\mathbf{C}$? Justify your answer.

7. Let $\mathbf{F}$ be any field and A, B any two 4×4-matrices over $\mathbf{F}$. Using the rational canonical form, prove that the following two conditions are equivalent:
(i) A and B are similar;
(ii) A and B have the same characteristic polynomial, the same minimal polynomial, and the same number of invariant factors.

8. Let V be a finite dimensional vector space over a field $\mathbf{F}$. A linear transformation $T : V \to V$ is called a *projection* if $T^2 = T$. Prove that two projections $T, S : V \to V$ are similar if and only if their kernels have the same dimension.

9. Find a splitting field $\mathbf{K}$ for $x^4 - 5$ over $\mathbf{Q}$, and determine $[\mathbf{K} : \mathbf{Q}]$.

10. Let $\mathbf{F}$ be an extension of degree 2002 of $\mathbf{Q}$. Prove that $\mathbf{F}$ contains no cubic roots of 2.