First Year Algebra Exam – May 2002

Time allowed: 240 minutes

Do **seven** of the following ten problems. Please do not turn in more than seven problems. The problems do **not** have to be attempted in the order they are listed.

You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, $\mathbf{Z}$, resp. $\mathbf{Q}$, $\mathbf{C}$, is the set of all integers, resp. rational numbers, complex numbers.

1. State and prove Eisenstein's criterion for polynomials over $\mathbf{Z}$. (You may use Gauss' lemma in your proof.)

2. Let G be a group of order $231 = 3 \cdot 7 \cdot 11$ and $Z(G)$ be the centre of G. Prove that a Sylow 7-subgroup of G is normal in G, and that $Z(G)$ contains a Sylow 11-subgroup of G.

3. Let G be a finite abelian group of order 2^n for some natural number n. Assume that G has exactly **one** element of order 2. Prove that G is cyclic. (**Hint :** You might want to use the structure theorem for finite abelian groups.)

4. (a) Define *Euclidean domain* and *principal ideal domain.*
(b) Prove that any Euclidean domain is a principal ideal domain.

5. Let $S = \mathbf{Z}[\sqrt{-5}]$ be the subring of $\mathbf{C}$ consisting of complex numbers of the form $a + b\sqrt{-5}$, $a, b \in \mathbf{Z}$. Show that S is **not** a unique factorization domain.

6. Let $\mathbf{F}$ be a field, and let $R = M_2(\mathbf{F})$ be the ring of all 2×2-matrices over $\mathbf{F}$. Let $V = \mathbf{F}^2$ be the vector space of all 2×1-matrices over $\mathbf{F}$. You may assume that V is a left R-module under the usual operations $u + w$, $x \cdot w$ for $u, w \in V$ and $x \in R$.
(a) Let $v \in V$ be an arbitrary nonzero element. Prove that $V = Rv$.
(b) Using (a), prove that the R-module V is irreducible. (An R-module M is called *irreducible* if $M \neq 0$ and $\{0\}$ and M are the only submodules of M.)

7. Let T and S be linear transformations $\mathbf{C}^4 \to \mathbf{C}^4$, both with the same characteristic polynomial $x^2(x-2)(x-4)$. Assume that $Ker(T)$ and $Ker(S)$ have the same dimension. Are T and S necessarily similar? Justify your answer.

8. Let $\mathbf{F}$ be a field of characteristic **3**. How many conjugacy classes of elements of order **3** are there in the group $GL_4(\mathbf{F})$ of invertible 4×4-matrices over $\mathbf{F}$? Justify your answer and give a representative for each conjugacy class.

9. Find a splitting field $\mathbf{K}$ for $x^7 - 3$ over $\mathbf{Q}$, and determine $[\mathbf{K} : \mathbf{Q}]$.

10. Let $\overline{\mathbf{Q}}$ be a subfield of $\mathbf{C}$ consisting of all complex numbers that are algebraic over $\mathbf{Q}$. Prove that $\overline{\mathbf{Q}}$ is **not** a finite extension of $\mathbf{Q}$.