First Year Algebra Exam – January 2002

Time allowed: 240 minutes

Do seven of the following ten problems. Please do not turn in more than seven problems. You must show your work. Answers with no work and/or no explanations will receive no credit. State clearly any theorem you use in your proofs.

In the problems, **Z**, resp. **Q**, **C**, is the set of all integers, resp. rational numbers, complex numbers.

1. State and prove Cayley's Theorem about finite groups.

2. Let H be a normal subgroup of G. Assume H is cyclic. Prove that any subgroup of H is also normal in G.

3. Prove that there is no simple group of order $2002 = 2 \times 7 \times 11 \times 13$.

4. Consider the ring $\mathbf{Z}[x]$ of polynomials in one variable x over the integers. Is $\mathbf{Z}[x]$ (i) a Euclidean domain ? (ii) a principal ideal domain ? (iii) a unique factorization domain ? Justify your answers.

5. Let R be a ring. An R-module M is called *irreducible* if $M \neq 0$ and 0 and M are the only submodules of M.

a) Prove Schur's Lemma: *Suppose the R-modules M and N are irreducible. Then every nonzero homomorphism $T : M \to N$ is an isomorphism.*

b) Let M be an irreducible R-module. Deduce from a) that $End_R(M)$, the set of all (module) homomorphisms $M \to M$, is a division ring.

6. Let $\mathbf{F}$ be a field and G be a finite subgroup of $\mathbf{F} \setminus \{0\}$, the multiplicative group of $\mathbf{F}$. Prove that G is cyclic.

7. Let A and B be two 4×4 matrices over $\mathbf{C}$, both with the same characteristic polynomial $(x-1)^2(x-2)(x-3)$. Assume that A and B have the same minimal polynomial. Are A and B necessarily similar? Justify your answer.

8. How many conjugacy classes of elements of order 4 are there in the group $GL_3(\mathbf{Q})$ of invertible 3×3-matrices over $\mathbf{Q}$? Justify your answer and give a representative for each conjugacy class.

9. Find a splitting field $\mathbf{F}$ for $x^6 - 4$ over $\mathbf{Q}$, and find $[\mathbf{F} : \mathbf{Q}]$.

10. Let $\mathbf{F}$ be a field and $\mathbf{E}$ be an extension of $\mathbf{F}$ of degree 2001. Prove that $\mathbf{F}(u) = \mathbf{F}(u^2)$ for any $u \in \mathbf{E}$.