FIRST YEAR ALGEBRA EXAMINATION

May 17, 2001

6:00 p.m. - 10:00 p.m.

Do seven of the following ten questions (the exam committee will NOT select your best seven answers if you answer more than seven). You may quote standard results (within reason) as long as you make it clear that are doing so and you state them clearly.

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1. State the three Sylow theorems and prove the existence of Sylow subgroups.

2. Let S_n denote the symmetric group of degree n. Let H be a subgroup of S_n. Prove that either all the elements of H are even permutations, or else exactly half of the elements of H are even permutations.

3. Prove that a group of order 280 is not simple.

4. Prove that if R is a non-zero commutative ring with identity, then R has a maximal ideal. (You may assume Zorn's Lemma).

5. Prove that, if n is any prime number, then the polynomial $\Phi_n(x) = x^{n-1} + x^{n-2} + \cdots + 1 \in \mathbf{Q}[x]$ is irreducible. (You may NOT assume that cyclotomic polynomials are irreducible).

6. Factor the number $2 \cdot 7^2 \cdot 17$ into irreducibles in the ring of Gaussian integers $\mathbf{Z}[\mathbf{i}]$. Prove that each of the elements you use is actually irreducible in $\mathbf{Z}[\mathbf{i}]$.

7. Let M be a module over R, an integral domain. Define what is the *torsion part* of M and prove that it is a submodule of M.

8. Let T and S be linear transformations $\mathbf{C}^5 \to \mathbf{C}^5$ over $\mathbf{C}$ both with characteristic polynomial $(x-2)^3(x-4)(x-5)$. Are T and S necessarily similar? Justify your answer.

9. How many conjugacy classes of elements of order 5 are there in the group $GL(8, \mathbf{Q})$ of 8 by 8 invertible matrices over $\mathbf{Q}$? Justify your answer and give a representative for each conjugacy class.

10. Prove that if q is a power of a prime, and $q > 1$, then there exists a field with exactly q elements. Prove furthermore that all fields of order q are isomorphic.