FIRST YEAR ALGEBRA EXAMINATION

January 16, 2001

6:00 p.m. - 10:00 p.m.

Do seven of the following ten questions (the exam committee will NOT select your best seven answers if you answer more than seven). You may quote standard results (within reason) as long as you make it clear that are doing so and you state them clearly.

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1. Let G be a group of order $5^2 \cdot 11$. Prove that:
   (1) G has a cyclic normal subgroup of order 55.
   (2) Assuming G is not abelian, show that the center of G is cyclic of order 5.

2. Let $G = \mathbf{Z}_{p^2} \times \mathbf{Z}_{p^2}$ be a group which is the direct product of two cyclic groups of order p^2, where p is some prime. Find the number of subgroups of order p^2 of G.

3. Prove that if G is a group of order p^α, where p is some prime and $\alpha \geq 1$, then the center of G is not trivial.

4. State and prove Eisenstein's Criterion for irreducibility.

5. Let R be a commutative ring with 1. Let I and J be ideals of R such that $I + J = R$. Prove that $R/(I \cap J)$ is isomorphic, as a ring, to $R/J \times R/I$.

6. Let $p(x) = x^4 + 3x^2 - 9x + 15 \in \mathbf{Q}[x]$. Prove that $R = \mathbf{Q}[x]/(p(x))$ is a field, and calculate the dimension of R as a vector space over $\mathbf{Q}$.

7. Prove that if F is a finite field, then $|F| = p^n$ for some prime p, and F has a subfield isomorphic to $\mathbf{Z}/p\mathbf{Z}$.

8. Find a representative for each conjugacy class of elements of order 3 in the group $\mathrm{GL}_4(2)$ of invertible 4×4 matrices over the field of integers modulo 2.

9. Let M be a module over a ring R with identity.
   (1) Define what is meant for M to be *irreducible.*
   (2) Prove Schur's Lemma that if M is irreducible, then every non-zero endomorphism of M is an automorphism.
   (3) Give an example of a ring R, an irreducible module M and some non-zero non-identity automorphism of M.

10. Let $T : \mathbf{C}^6 \to \mathbf{C}^6$ be a $\mathbf{C}$-linear transformation with minimum polynomial $(x-2)^2(x+2)^3$. Find the possible Jordan canonical forms for T.