First Year Algebra Exam – September 2000

Time allowed: 240 minutes

Do seven of the following ten problems. Please do not turn in more than seven problems. You must show your work. Answers with no work and/or no explanations will receive no credit. State clearly any theorem you use in your proofs.

In the problems, $\mathbf{Q}$ is the field of all rational numbers.

1. Let G be a group of order 429. Prove that the Sylow 13-subgroups of G are normal, and that any Sylow 11-subgroup is contained in the center of G.

2. Let p be a prime and G a non-abelian group of order p^3. Find the order of the center of G.

3. Classify (up to isomorphism) all finite groups with exactly two conjugacy classes.

4. a) Define *Euclidean domain* and *principal ideal domain*.

b) Prove that any Euclidean domain is a principal ideal domain.

5. Let R be a ring. An element $x \in R$ is said to be *nilpotent*, if there is a natural number n such that $x^n = 0$. Let $N(R)$ be the set of all nilpotent elements of R.

a) Suppose the ring R is commutative. Prove that $N(R)$ is an ideal of R.

b) Is the statement "$N(R)$ is a left ideal of R" still true, if R is not commutative ? Justify your answer.

6. Let R be the set of all sequences $(a_1, a_2, a_3, \ldots)$ of integers $a_1, a_2, a_3, \ldots$ where all but finitely many of the a_i are 0. You may assume that R is a ring under componentwise addition and multiplication:

$$(a_1, a_2, a_3, \ldots) + (b_1, b_2, b_3, \ldots) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots),$$
$$(a_1, a_2, a_3, \ldots) \cdot (b_1, b_2, b_3, \ldots) = (a_1 b_1, a_2 b_2, a_3 b_3, \ldots).$$

Prove that

a) R does not have any identity element for multiplication.

b) R considered as a module over itself is not finitely generated.

7. For a field $\mathbf{F}$, let $GL_4(\mathbf{F})$ be the group of all invertible 4×4-matrices over $\mathbf{F}$. Using rational canonical form, find all conjugacy classes (and one representative for each class) of elements of order 5 in

a) $GL_4(\mathbf{Q})$;

b) $GL_4(\mathbf{F})$, where $\mathbf{F}$ is a field of characteristic 5.

8. Let p be a prime and $\mathbf{F}$ be a field with p elements. Let $G = GL_2(\mathbf{F})$ be the group of all invertible 2×2-matrices over $\mathbf{F}$.

a) Using rational canonical form (or Jordan canonical form), show that G has exactly one conjugacy class of elements of order p.

b) Given that $|G| = p(p-1)(p^2-1)$, find the total number of elements of order p in G.

9. Find a splitting field $\mathbf{F}$ for $x^4 - 2x^2 - 2$ over $\mathbf{Q}$, and determine $[\mathbf{F} : \mathbf{Q}]$.

10. Let $\mathbf{F} = \mathbf{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{2000})$ be the field obtained from $\mathbf{Q}$ by joining all the numbers $\sqrt{n}$, $2 \leq n \leq 2000$. Prove that $\sqrt[3]{3} \notin \mathbf{F}$.