

First Year Algebra Exam – May 2000

Time allowed: 240 minutes

Do SEVEN of the following ten problems. Please do not turn in more than seven problems.

You must show your work. Answers with no work and/or no explanations will receive NO credit. State clearly any theorem you use in your proofs.

In the problems, \mathbf{Z} , resp. \mathbf{Q} , \mathbf{C} , is the set of all integers, resp. all rational, all complex, numbers.

1. Classify (up to isomorphism) groups of order 39.

2. Prove that there is no simple group of order 80. (Caution: You are not allowed to use the $p^a q^b$ -theorem of Burnside !)

3. State and prove Eisenstein's criterion for polynomials over \mathbf{Z} . (In the proof you may use Gauss' lemma).

4. Let p be a prime. Define

$$T = \{x = \frac{a}{b} \in \mathbf{Q} \mid a, b \in \mathbf{Z}, \gcd(b, p) = 1\},$$

$$I = \{x = \frac{a}{b} \in \mathbf{Q} \mid a, b \in \mathbf{Z}, \gcd(b, p) = 1, p \mid a\}.$$

Show that (i) T is a subring of \mathbf{Q} , (ii) I is a maximal two-sided ideal of T , (iii) T/I is isomorphic to the field $\mathbf{Z}/p\mathbf{Z}$ of p elements.

5. Let I be the smallest ideal of the ring $S = \mathbf{Z}[\sqrt{-5}]$ that contains 2 and $1 + \sqrt{-5}$. Prove that I is not a principal ideal of S .

6. For a field \mathbf{F} , let $GL_3(\mathbf{F})$ be the group of all invertible 3×3 -matrices over \mathbf{F} under matrix multiplication. Using rational canonical form, find all conjugacy classes (and one representative for each class) of elements of order 4 in

a) $GL_3(\mathbf{Q})$;

b) $GL_3(\mathbf{F})$, where \mathbf{F} is a field of characteristic 2.

7. Let \mathbf{F} be any field and A, B any two 4×4 -matrices over \mathbf{F} . Using rational canonical form, prove that the following two conditions are equivalent:

(i) A and B are similar;

(ii) A and B have the same characteristic polynomial, the same minimal polynomial, and the same number of invariant factors.

8. Let n be a natural number.

a) Using Eisenstein's criterion, prove that there is at least one irreducible polynomial of degree n over \mathbf{Q} .

b) Using a), prove that there is at least one extension of \mathbf{Q} of degree n .

Are the statements still true if one replaces \mathbf{Q} by \mathbf{C} ?

9. Find a splitting field \mathbf{F} for $x^6 - 9$ over \mathbf{Q} , and determine $[\mathbf{F} : \mathbf{Q}]$.

10. Let p and q be primes and let $\alpha = \sqrt{p} + \sqrt[3]{q}$. Is α algebraic over \mathbf{Q} , and if it is, find the degree of α over \mathbf{Q} .