First–Year Algebra Examination, January 2000

Instructions: Do seven of the following ten exercises. Please do not hand in more than seven.

1. (a) Compute the order of the general linear group $\mathrm{GL}_n(\mathbb{Z}_p)$, where p is a prime number.

   (b) Calculate the order of the subgroup $\mathrm{SL}_n(\mathbb{Z}_p)$ of matrices of determinant one.

2. Let $p < q$ be distinct prime numbers.

   (a) Show that there are at most two nonisomorphic groups of order pq.

   (b) Specify conditions on p and q under which there is only one such group, up to isomorphism, and show that it is cyclic.

   (c) In the remaining cases, "construct" or otherwise describe the other group, and indicate why it is non–commutative.

3. (a) Prove that S_4 contains no nonabelian simple groups.

   (b) Use (a) to prove that if G is a nonabelian simple group, then every proper subgroup of G has index at least 5.

4. Suppose that R is a commutative ring. Prove that the modular law holds for ideals of R; that is, show that for every trio of ideals I, J and K of R for which $I \subseteq K$ we have

$$I + (J \cap K) = (I + J) \cap K.$$

5. Suppose that R is a commutative ring with an identity. Show, using Zorn's Lemma, that every ideal of R is contained in a maximal ideal.

6. A ring R is *boolean* if $x^2 = x$, for each $x \in R$.

   (a) Show that any boolean ring is commutative and has characteristic 2.

   (b) Use the Chinese Remainder Theorem to prove that every finite boolean ring has 2^n elements, for a suitable positive integer n.

7. Prove the following form of Eisenstein's Criterion for irreducibility of polynomials: *Suppose that R is an integral domain and P is a prime ideal of R. Let*

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with coefficients in R, such that $a_o, a_1, \ldots, a_{n-1} \in P$, but $a_o \notin P^2$. Then $f(x)$ is irreducible over R.

8. Prove that in $GL_2(\mathbb{Q})$ all the elements of order four are conjugate.

9. Prove, over any field F, that if two 2×2 matrices or two 3×3 matrices have the same minimum and characteristic polynomials then they are similar matrices.

   Give an example which shows that this is false for matrices of greater dimension.

10. Suppose that F is a field whose characteristic is not 2. Assume that $d_1, d_2 \in F$ are not squares in F. Prove that $F(\sqrt{d_1}, \sqrt{d_2})$ is of dimension 4 over F if $d_1 d_2$ is not a square in F and of dimension 2 otherwise.