First Year Algebra Examination; September, 1999

Work 7 out of the 10 exercises below. Please do not turn in more than 7. They need not be handed in in the order in which they appear here. When referring to or quoting theorems please do so with precision and clarity.

1. State and prove Cayley's Theorem about finite groups.

2. (a) Prove that S_4 contains no non–abelian simple groups.

   (b) Use the result of (a) to prove that if G is a nonabelian simple group, then every proper subgroup of G has index at least 5.

3. Describe all the groups of 76 elements, up to isomorphism. (Hint: use semidirect products.)

4. Using Zorn's Lemma, prove that in each commutative ring with identity minimal prime ideals exist.

5. (a) Define *Euclidean domain* and *principal ideal domain.*

   (b) Prove that any Euclidean domain is a principal ideal domain.

6. Prove that $X^2 + Y^2 - 1$ is irreducible in $\mathbb{Q}[X, Y]$. (Hint: Translate by a suitable quantity and then apply the general form of Eisenstein's Criterion.)

7. Let R be a ring with identity. Suppose that $\phi : M \longrightarrow F$ is a surjective R-homomorphism and that F is a free R-module. Prove that $M = \ker(\phi) \oplus N$, where $N \cong F$.

8. Prove that in $\mathrm{GL}_2(\mathbb{Q})$ all the elements of order four are conjugate. (Hint: Consider the rational canonical form of such an element.)

9. A *projection* is a linear transformation $P : V \longrightarrow V$ on a vector space V for which $P^2 = P$. Assume that V has finite dimension and prove the following:

   (a) Any projection is diagonalizable.

   (b) Two projections have the same diagonal form if and only if their kernels have the same dimension.

10. Determine the dimension over $\mathbb{Q}$ of the extension $\mathbb{Q}(\sqrt{3 + 2\sqrt{2}})$. Justify your arguments.