

Algebra First Year Examination  
May, 1999

Work on seven out of the following ten exercises. Please do not turn in more than seven solutions. When using results quote them explicitly and carefully; and when in doubt whether it is legitimate to use a theorem at least sketch a proof. You have four hours for this examination.

1. Let  $p$  be a prime number. Recall that a group  $G$  is called a  $p$ -group if for each  $1 \neq g \in G$ ,  $g^{p^i} = 1$ , for some positive integer  $i$ . Prove the following about a *nontrivial*  $p$ -group  $G$ :
  - (a)  $|G| = p^n$ , for some positive integer  $n$ . (3 points)
  - (b)  $G$  has a nontrivial center. (4 points)
  - (c)  $G$  is nilpotent. (3 points)
  
2. Suppose that  $G$  is a group of order 165.
  - (a) Prove that  $G$  has a normal Sylow 11-subgroup  $H$ . (4 points)
  - (b) Show that  $H$  either lies in the center of  $G$ , or else  $G$  has a normal subgroup of order 33. (6 points)
  
3. Prove that  $A_5$  is a simple group.
  
4. Let  $A$  be an integral domain, and  $P$  be a prime ideal of  $A$ . Define  $A_P$  to be the subset of the quotient field  $K$  of  $A$ , consisting of all fractions whose denominator is not in  $P$ . Prove that
  - (a)  $A_P$  is a subring of  $K$ .
  - (b)  $A_P$  has exactly one maximal ideal; identify it.
  
5. Let  $n$  be a natural number; prove that the polynomial

$$\Phi_n(X) = \frac{X^n - 1}{X - 1}$$

is irreducible over the ring  $\mathbf{Z}$  if and only if  $n$  is prime.

6. Suppose that  $V$  is a finite dimensional vector space over the field  $F$  and that  $T : V \rightarrow W$  is a linear transformation into a vector space  $W$  over  $F$ . Prove that

$$\dim(V) = \dim(\ker(T)) + \dim(\text{im}(T)).$$

(Caution: The finite dimensionality of  $\text{im}(T)$  must be established.)

7. Let  $R$  be a ring with identity. Suppose that  $\phi : M \rightarrow F$  is a surjective  $R$ -homomorphism and that  $F$  is a free  $R$ -module. Prove that  $M = \ker(\phi) \oplus N$ , where  $N$  is a submodule of  $M$  isomorphic to  $F$ .

8. Let  $A$  be the  $n \times n$  matrix with real entries defined by

$$A_{ij} = \begin{cases} 0 & \text{for all } i \text{ and } j, \text{ except when } i = j + 1 \\ 1 & \text{when } i = j + 1 \end{cases}.$$

Prove that  $A$  has no square root in the ring  $M_n(\mathbb{R})$  of all  $n \times n$  matrices over  $\mathbb{R}$ .

9. This exercise concerns  $G = \text{SL}_2(\mathbb{F}_3)$ , the group of invertible matrices of determinant 1 over the field of 3 elements. Prove, using the rational canonical forms, that:
- (a) there is only one matrix in  $G$  of order 2 and it lies in the center of  $G$ ; (3 points)
  - (b) there are no elements of order 8 in  $G$ ; (4 points)
  - (c) regard  $G$  as a subgroup of  $\text{SL}_2(\mathbb{F}_9)$ , and show each element of  $G$  of order 4 is conjugate to a diagonal matrix. ( $\mathbb{F}_9$  stands for the field of 9 elements.) (3 points)
10. Over the field  $\mathbb{Q}$  of rational numbers, consider the polynomial  $q(X) = X^p - 2$ , with  $p$  prime. Prove that the splitting field of  $q(X)$  is  $L = \mathbb{Q}(\sqrt[p]{2}, \zeta)$ , where  $\zeta$  is a primitive  $p$ -th root of 1, and convince that  $[L : \mathbb{Q}] = p(p - 1)$ .