MASTERS' EXAMINATION IN ALGEBRA (MAY 1998)

TIME ALLOWED: four HOURS

ANSWER seven QUESTIONS. IF YOU ATTEMPT MORE THAN SEVEN, INDICATE THE SEVEN TO BE GRADED.

Read the questions very carefully. **State clearly any theorems you use in your proofs.**

1.(a) (3 points) State the class equation for finite groups.
(b) (4 points) Prove that a group of prime power order p^a, $(a > 1)$ has a nontrivial center.
(c) (3 points) Hence or otherwise prove that such a group has normal subgroups of orders p^b for all $0 \le b \le a$. (Hint: Factor out a normal subgroup and use induction)

2. Let F be a field.
(a) (3 points) Define the *minimal polynomial* of an element in some extension field of F, and the *minimal polynomial* of a linear transformation of a finite-dimensional vector space over F.
(b) (7 points) Show that the former must always be irreducible, but the latter need not be.

3. (a) (5 points) Using the ring of polynomial $\mathbb{F}_5[X]$ with coefficients in the field of integers mod 5, or otherwise, prove the existence of a field F of 125 elements.
(b) (5 points) Show that the elements of F are precisely the roots of the polynomial $X^{125} - X \in \mathbb{F}_5[X]$.

4. (10 points) Let $F \subseteq K$ be a field extension and $\alpha \in K$ Prove the equivalence of the following statements:

(a) α is algebraic over F.
(b) $F(\alpha) = F[\alpha]$.
(c) $F(\alpha)$ has finite degree over F.

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5. (a) (5 points) Prove that every ideal in a Euclidean ring is principal.
(b) (5 points) Give an example of an ideal in a unique factorization domain which is not principal. Justify your answer.

6. (10 points) Find one representative of each conjugacy class of elements of order 3 in the group $GL(5, \mathbb{F}_3)$ of invertible 5×5 matrices with entries in the field of integers mod 3. Hint: $x^3 - 1 = (x-1)^3$.

7. Let R be a ring with unity 1, M a (unital) left R-module and $E(M)$ the set of R-module homomorphisms from M to M.
(a) (5 points) Show how to give $E(M)$ the structure of a ring in which multiplication is composition of homomorphisms.
(b) (5 points) Prove Schur's Lemma: If $M \neq \{0\}$ and has no submodules other than $\{0\}$ and M, then $E(M)$ is a division ring.

8. (a) (6 points) Prove that a group of order 1998 has a normal subgroup of order 37 and another normal subgroup of index 2.
(b) (4 points) Show further that any such group has an element of order 111.

9. (10 points) Show that the ring of $n \times n$ matrices over a field has no two-sided ideals other than the zero ideal and the ring itself. Exhibit a nonzero proper, left ideal of this ring.

10. (10 points) State and prove Cayley's theorem about finite groups.