First Year Algebra Exam (8/28/97) **Time allowed: Four hours**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let R be a commutative ring and let I be an ideal in R. Prove that $I[X]$ is an ideal in $R[X]$ and that $R[X]/I[X] \cong (R/I)[X]$.

2. State and prove Eisenstein's criterion for the irreducibility of a monic polynomial $f(X) \in \mathbb{Z}[X]$

3. Let $k \leq n$ and let $\sigma = (1\ 2 \ldots k)$ be a k-cycle in the symmetric group S_n. Describe the conjugacy class of σ and the centralizer of σ explicitly. Justify your answers.

4. Let R be a commutative ring with 1.

   (a) Let S be a set and let $F(S)$ be the free R-module on S. State the universal mapping theorem for $F(S)$.

   (b) Prove that for any R-module M there is a free R-module F and an onto R-module homomorphism $\phi : F \to M$.

5. Find a single representative for each conjugacy class of elements of order 4 in $GL_2(\mathbb{Z}/5\mathbb{Z})$.

6. Using Zorn's Lemma, prove that in each commutative ring with identity minimal prime ideals exist.

7. Let G be a group of 385 elements. Prove that the Sylow 11-subgroups are normal, and that any Sylow 7-subgroup lies in the center.

8. Let p be prime and let Z_{p^2} be the cyclic group of order p^2. Prove that the automorphism group of Z_{p^2} is cyclic.

9. Let R be a ring with identity and M be an R-module. An element $x \in M$ is called a *torsion element* if $rx = 0$, for some nonzero $r \in R$. Let $T(M)$ denote the subset of all torsion elements of M.

   (a) If R is an integral domain show that $T(M)$ is a submodule of M.

   (b) Give an example to show that $T(M)$, in general, is not a submodule of M.

10. Suppose that V is a finite dimensional vector space over the field F and that $T : V \longrightarrow W$ is a linear transformation into a vector space W over F. Prove that $\dim(V) = \dim(\ker(T)) + \dim(\operatorname{Im}(T))$. (Caution: The finite dimensionality of $\operatorname{Im}(T)$ must be established.)