First Year Algebra Exam (5/22/97) **Time allowed: Four hours**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Prove that a group of order 30 must have a normal subgroup of order 15.

2. Prove that D_{2n} is nilpotent if and only if n is a power of 2. (Hint: Use the upper central series; show that if $n \geq 3$ then $Z(D_{2n}) \neq \{1\}$ if and only if n is even.)

3. Let $\mathbb{C}$ be the field of complex numbers. Prove that each irreducible $\mathbb{C}[X]$-module is isomorphic to $\mathbb{C}$.

4. Suppose that A is a commutative ring with identity, and I is an ideal of A.

   (a) For each positive integer n, prove that

   $$A^n/IA^n \cong A/I \times \cdots \times A/I.$$

   (b) Use (a) to prove that if $A^m \cong A^n$, where m and n are positive integers, then $m = n$. (You may use the corresponding fact for fields.)

5. Prove that $X^2 + Y^2 - 1$ is irreducible in $\mathbb{Q}[X, Y]$. (Hint: Translate by a suitable quantity and then apply the general form of Eisenstein's Criterion.)

6. Let R be a UFD.

   (a) Define what it means for $f(X) \in R[X]$ to be a primitive polynomial.

   (b) Let $f, g \in R[X]$. Prove that fg is primitive if and only if f and g are both primitive.

7. Let V be a vector space over a field F. Prove that V has a basis. (Do not assume that V is finitely generated.)

8. Find a representative for every conjugacy class of elements of order 5 in $GL_8(\mathbb{Q})$.

9. Determine all homomorphisms $\phi : Z_3 \to \mathrm{Aut}(Z_9)$ and construct the associated semidirect products. Prove that the semidirect products associated to the non-trivial homomorphisms are all isomorphic.

10. Prove that every field F contains a unique smallest subfield F_0, and that F_0 is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime p.