First Year Algebra Examination; January, 1997

Do seven of the following ten exercises. Please do not turn in more than seven exercises.

1. (a) Prove that there are exactly four homomorphisms from $\mathbb{Z}_2$ into $Aut(\mathbb{Z}_8)$.

   (b) Show that these yield four pairwise nonisomorphic semidirect products.

2. State and prove the Orbit-Stabilizer Theorem.

3. Suppose that $|G| = 105$. If G has a normal Sylow 3-subgroup, prove that it must lie in the center of G, and that G is solvable.

4. Let F be a field, and $A = F[[X]]$ denote the ring of formal power series in one variable. Prove the following:

   (a) The units of A are precisely the power series whose constant term is nonzero.

   (b) Let I_k denote the set of all power series $\Sigma_{n=0}^{\infty} a_n x^n$ for which $a_0, \ldots, a_{k-1}$ are all zero. Prove that each I_k is an ideal of A, and that these are the only nonzero ideals of A.

5. Prove the Divison Algorithm for the ring $\mathbb{Z}[i]$ of Gaussian integers.

6. Let n be a natural number; prove that the polynomial

$$\Phi_n(X) = \frac{X^n - 1}{X - 1}$$

   is irreducible over the ring $\mathbb{Z}$ precisely when n is prime.

7. Suppose that $T : V \longrightarrow W$ is a linear transformation between vector spaces over the same field F. Prove that T is one to one precisely when it maps linearly independent sets to linearly independent sets.

8. Let R be a ring with identity. Suppose that $\phi : M \longrightarrow F$ is a surjective R-homomorphism and that F is a free R-module. Prove that $M = Ker(\phi) \oplus N$, where $N \cong F$.

9. Prove that in $GL_2(\mathbb{Q})$ all the elements of order four are conjugate. (Hint: Consider the rational canonical form of such an element.)

10. Let F be a field and suppose that K is a field extension of F. Prove that the set of elements of K which are algebraic over F form a subfield of K containing F.