# Unipotent elements, Standard Involutions
# and the Divisor Matrix

I set

$$J_n = I + \sum_{i=1}^{n-1} e_{i\ i+1} \in GL(n,\mathbb{Z}).$$

This unipotent element, the Jordan block of size $n$, occurs in many situations, as does $J_{n_1,n_2,\ldots,n_r} = J_{n_1} \oplus \cdots \oplus J_{n_r}$. There is an involution $T_n$ in $GL(n,\mathbb{Z})$ which inverts $J_n$ whose first row is $(10\ldots0)$; $T_n$ is uniquely determined by these conditions. Thus, $T_{n_1} \oplus \cdots \oplus T_{n_r}$ inverts $J_{n_1}, \ldots, _{n_r}$, and I call such an involution standard, and any conjugate of the standard involution by an element of $< J_{n_1} > \times \cdots \times < J_{n_r} >$ is also deemed to be standard.

By the divisor matrix, I mean the $\mathbb{N} \times \mathbb{N}$ matrix $D = (d_{ij}), (i,j) \in \mathbb{N} \times \mathbb{N}$, where

$$d_{ij} = \begin{cases} 1 \text{ if } i|j \\ 0 \text{ otherwise.} \end{cases}$$

This is an infinite, upper triangular matrix with 1 on the diagonal, and one checks that

$$D^{-1} = (\overline{d}_{ij}),$$

where

$$\overline{d}_{ij} = \begin{cases} \mu(\frac{j}{i}) \text{ if } i|j \\ 0 \text{ otherwise,} \end{cases}$$

$\mu$ being the Möbius function.

For each $n \in \mathbb{N}$, set $D_n = (d_{ij})_{\substack{1\le i\le n \\ 1\le j\le n}}$, the truncation of $D$ of level $n$. Each $D_n$ is a unipotent element of $GL(n,\mathbb{Z})$ and the preceding discussion produces a standard involution $T(n)$, characterized as the only involution in $GL(n, \mathbb{Z})$ which inverts $D_n$ and such that for each odd $i \leq n$, the $i^{th}$ row is $(\delta_{i1}\delta_{i2}\dots\delta_{in})$. Piecing these involutions together produces an involution $T$ inverting $D$ such that the truncation of $T$ of level $n$ is $T(n)$ for all $n \in \mathbb{N}$.

The object of this note is to describe $T$ rather explicitly, and to mention some of its properties. I write

$$T = (t_{ij}), (i,j) \in \mathbb{N} \times \mathbb{N}$$

and proceed to determine the $t_{ij}$. As already remarked,

$$t_{ij} = \delta_{ij} \text{ if } i \text{ is odd}$$

so we need to determine the $t_{ij}$ when $i$ is even.

In order to capture the $i^{th}$ row of $T$, I introduce a Dirichlet series $r_i$ by setting

$$r_i(s) = \sum_{n \in \mathbb{N}} \frac{t_{in}}{n^s}$$

At this point, $r_i$ is just a formal series, but obviously, $r_d(s) = d^{-s}$ for all odd $d$.

**Theorem.** *For all* $a \in \mathbb{N}$ *and all odd* $n \in \mathbb{N}$,

$$r_{2^a n}(s) = \frac{1}{n^s} r_{2^a}(s), \tag{1}$$

and

$$r_{2^a}(s) = (1 - (1 - 2^{-s})\zeta(s)^2)^a \tag{2}$$

where $\zeta$ is the Riemann zeta function.

*Proof.* If $n \in \mathbb{N}$ and $p$ is a prime, $p^{\nu_p(n)}$ is in the highest power of $p$ which divides $n$. Also, I set

$$\sigma_s(n) = \sum_{d|n} d^{-s}.$$

Thus, for example,

$$\sigma_0(n) = \prod_p (1 + \nu_p(n)), \tag{3}$$

and

$$\sum_{n \text{ odd}} \frac{\sigma_0(n)}{n^s} = \prod_{p>2} (1 - p^{-s})^{-2} \tag{4}$$

We examine the equation $TD = D^{-1}T$. Comparing $(i, j)$ entries gives

$$\sum_k t_{ik} d_{kj} = \sum_k \overline{d}_{ik} t_{kj},$$

that is

$$\sum_{k|j} t_{ik} = \sum_{i|k} \mu(\frac{k}{i}) t_{kj}. \tag{5}$$

In order to exploit (5), I interpolate two easy lemmas.

**Lemma 1.** *If* $t_{ij} \neq 0$ *and* $p$ *is an odd prime, then* $\nu_p(i) \leq \nu_p(j)$.

**Lemma 2.** *If* $t_{ij} \neq 0$*, then*

$$\sum_p \nu_p(i) \leq \sum_p \nu_p(j).$$

I prove both lemmas at the same time. Suppose one of the lemmas is false. Let $j$ be the smallest integer such that $t_{ij} \neq 0$ and either Lemma 1 or Lemma 2 is false for $(i,j)$. For this $j$, let $i$ be the largest integer such that $t_{ij} \neq 0$ and either Lemma 1 or Lemma 2 is false for $(i,j)$. Since $t_{ab} = \delta_{ab}$ for all odd $a$, and since obviously $i \neq j$, it follows that $i$ is even, $i = 2i_0$. We use (5) for the pair $(i_0, j)$. I argue that if $k|j$ and $k < j$, then $t_{i_0 k} = 0$. Suppose Lemma 1 fails for $(i,j)$ and the odd prime $p$. Since $\nu_p(i_o) = \nu_p(i)$, and $\nu_p(k) \leq \nu_p(j)$ for all divisors $k$ of $j$, and since $\nu_p(i) > \nu_p(j)$, we get $\nu_p(i_o) > \nu_p(k)$, and so our choice of $j$ forces $j$ forces $t_{i_o k} = 0$ for every proper divisor $k$ of $j$. Suppose Lemma 2 fails for $(i,j)$ and $k$ is a proper divisor of $j$. Since

$$\sum_p \nu_p(i) > \sum_p \nu_p(j) > \sum_p \nu_p(k),$$

and since $\sum_p \nu_p(i_o) = -1 + \sum_p \nu_p(i)$ we have

$$\sum_p \nu_p(i_0) > \sum_p \nu_p(k).$$

Again, the minimality of $j$ forces $t_{i_0 k} = 0$. Hence

$$\sum_{k|j} t_{i_0 k} = t_{i_0 j}.$$

Next, I argue that if $i_o|k$ and $k > i$, then $t_{kj} = 0$. Suppose $t_{kj} \neq 0$. Since $i_0|k$, it follows that for all odd primes $p$, $\nu_p(k) \geq \nu_p(i_0)$, and as before $\nu_p(i_0) = \nu_p(i)$. Thus, if Lemma 1 fails for $(i,j)$ and $p$, it also fails for $(k,j)$ and $p$, against $k > i$ and the maximality of $i$. If Lemma 2 fails for $(i,j)$, then we get

$$\sum_p \nu_p(k) \geq 1 + \sum_p \nu_p(i_0) = \sum_p \nu_p(i) > \sum_p \nu_p(j),$$

so Lemma 2 fails for $(k,j)$ against maximality of $i$. Hence, we have shown that

$$\sum_{i_0|k} \mu(\frac{k}{i_0}) t_{kj} = \mu(1) t_{i_0 j} + \mu(2) t_{ij}.$$

By (5), we get

$$t_{i_o j} = \mu(1) t_{i_o j} + \mu(2) t_{ij},$$

against $\mu(1) = 1, \mu(2) = -1$. So Lemmas 1 and 2 hold.

If $(i, j) \in \mathbb{N} \times \mathbb{N}$, I set

$$\begin{aligned} L(i,j) &= \sum_{k|j} t_{ik}, \\ R(i,j) &= \sum_{m} \mu(m) t_{im,\ j}, \\ \text{so that } R(i,j) &= \sum_{i|k} \mu(\frac{k}{i}) t_{kj}, \text{ and} \\ L(i,j) &= R(i,j), \qquad (i,j) \in \mathbb{N} \times \mathbb{N}. \end{aligned}$$

Suppose now that $i$ is even and

$$i = 2^{a+1} d,\ a \in \mathbb{N}_0,\ d \text{ odd}.$$

Set

$$i_0 = 2^a d.$$

Concerning $i_0$, I insert another result.

**Lemma 3.** *If $j \in \mathbb{N}$ and $d \nmid j$, then $t_{i_0 m} = 0$ for every divisor $m$ of $j$ and $t_{kj} = 0$ for every multiple $k$ of $i_0$.*

*Proof.* This lemma follows from Lemma 1, for if $d \nmid j$, then there is an odd prime $p$ such that $\nu_p(d) > \nu_p(j)$, and for this prime, $\nu_p(i_o) > \nu_p(m)$ for every divisor $m$ of $j$ and $\nu_p(k) > \nu_p(j)$ for every multiple of $k$ of $i_o$.

Thus, we are led to examine $L(2^a d, 2^c dn)$ and $R(2^a d, 2^c dn)$, where $a, c \in \mathbb{N}_0$ and

$d$ and $n$ are odd. Applying Lemma 1, we get

$$L(2^a d, 2^c dn) = \sum_{b \le c} \sum_{m|n} t_{2^a d, 2^b dm}$$

$$R(2^a d, 2^c dn) = \sum_{m|n} \mu(m) t_{i_0 m, 2^c dn}$$

$$= \sum_{m|n} \mu(m) t_{i_0 m, 2^c dn} + \mu(2) \sum_{m|n} \mu(m) t_{im, 2^c dn}.$$

Since $L(2^a d, 2^c dn) = R(2^a d, 2^c dn)$ and since $\mu(2) = -1$, we get

$$\sum_{m|n} \mu(m) t_{2^{a+1} dm, 2^c dn} = \sum_{m|n} \mu(m) t_{2^a dm, 2^c dn} - \sum_{b \le c} \sum_{m|n} t_{2^a d, 2^b dm}. \tag{6}$$

For each divisor $h$ of $n$, let $(dh, \frac{n}{h})$ play the role of $(d, n)$ in (6):

$$\sum_{m|\frac{n}{h}} \mu(m) t_{2^{a+1} dhm, 2^c dn} = \sum_{m|\frac{n}{h}} \mu(m) t_{2^a dhm, 2^c dn} - \sum_{b \le c} \sum_{m|\frac{n}{h}} t_{2^a dh, 2^b dhm}, \tag{6.h}$$

where I have used the equality $dh \cdot \frac{n}{h} = dn$. Adding these equations $(6.h)$, as $h$ ranges over the divisors of $n$, yields

$$t_{2^{a+1} d, 2^c dn} = t_{2^a d, 2^c dn} - \sum_{h|n} \sum_{b \le c} \sum_{m|\frac{n}{h}} t_{2^a dh, 2^b dhm} \tag{7}$$

Here I have used the equations $\delta_{1k} = \sum_{l|k} \mu(\ell)$.

Suppose now that (1) holds for $a$. Then (7) gives

$$t_{2^{a+1} d, 2^c dn} = t_{2^a, 2^c n} \tag{7'}$$

$$-\sum_{h|n}\sum_{b\le c}\sum_{m|\frac{n}{h}} t_{2^a,2^b m}.$$

But (7) holds for all odd $d, n$, and in particular, for $d=1$, whence $(7)'$ forces (1) to hold for $a+1$:

$$t_{2^{a+1},2^c n} = t_{2^{a+1}d,2^c dn},$$

and then $(7)'$ reads

$$t_{2^{a+1},2^c n} = t_{2^a,2^c n}$$

$$-\sum_{h|n}\sum_{b\le c}\sum_{m|\frac{n}{h}} t_{2^a,2^b m}. \tag{8}$$

By inspection.

$$\sum_{h|n}\sum_{b\le c}\sum_{m|\frac{n}{h}} t_{2^a,2^b m} =$$

$$\sum_{b\le c}\sum_{m|n} t_{2^a,2^b m}\cdot \sigma_0(\frac{n}{m}),$$

so (8) becomes

$$t_{2^{a+1},2^c n} = t_{2^a,2^c n} - \sum_{b\le c}\sum_{m|n} t_{2^a,2^b m}\sigma_0(\frac{n}{m}). \tag{9}$$

This equation holds for all odd $n$ and all $c\in\mathbb{N}_0$, and so

$$\begin{aligned} r_{2^{a+1}}(s) =& r_{2^a}(s) - \sum_{c\ge 0}\sum_{n \text{ odd}}\sum_{b\le c}\sum_{m|n}\frac{t_{2^a,2^b m}}{(2^c n)^s}\sigma_0(\frac{n}{m}) \\ &= r_{2^a}(s) - \sum_{c-b\ge 0}\frac{1}{2^{(c-b)s}}\sum_{m \text{ odd}}\sum_{b\ge 0}\frac{t_{2^a,2^b m}}{(2^b m)^s}\cdot\sum_{\ell \text{odd}}\frac{\sigma(\ell)}{\ell^s} \\ &= r_{2^a}(s) - (1-2^{-s})^{-1} r_{2^a}(s)\cdot\prod_{p>2}(1-p^{-s})^{-2} \\ &= r_{2^a}(s)(1-(1-2^{-s})\zeta(s)^2), \end{aligned}$$

and the Theorem holds by induction on $a$.

I conclude with a remark. Let $T(n)$ be the truncation of $T$ of level $n$, so that

$$T(n) = (t_{ij})_{\substack{1\le i\le n,\\ 1\le j\le n}}$$

and set

$$\text{Sum } T(n) = \sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} t_{ij}$$

be the sum of all the entries of $T(n)$. Set

$$\text{Row Sum }_i T(n) = \sum_{j=1}^{n} t_{ij},$$

$$\text{Col Sum }_j T(n) = \sum_{i=1}^{n} t_{ij}.$$

Obviously,

$$\text{Sum } T(n) = \sum_{i=1}^{n} \text{ Row Sum}_i T(n) = \sum_{j=1}^{n} \text{ Col Sum}_j T(n).$$

I argue that $\text{Col Sum}_j T(n) = \mu(j)$. Namely, since $T$ is an involution and $TD = D^{-1}T$, so also $DT = TD^{-1}$. Since $d_{1k} = 1$ for all $k$, it follows that the $(1, j)$ entry of $DT$ is $\text{Col Sum}_j T(n)$, for all $j \le n$. Since $t_{1k} = \delta_{1k}$, the $(i, j)$ entry of $TD^{-1}$ is $\overline{d}_{1j} = \mu(\frac{j}{1}) = \mu(j)$.

So

$$(*) \qquad M(n) = \textstyle\sum_{j=1}^{n} \mu(j) = \sum_{i=1}^{n} \text{ Row Sum}_i T(n),$$

where $M$ is the summatory function for $\zeta^{-1}$. I think of $(*)$ as expressing the incomprehensible by the inscrutable.

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