Philosophy and Mathematics: *A Gödelian Connection*

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Abstract

The purpose of this paper is to indicate some connections between Gödel's philosophical convictions with respect to the foundations of mathematics, and his mathematical discoveries of 1931. We claim that Gödel's Platonistic view of mathematics was a determinant factor towards his discovery of the incompleteness theorems. In section 2, we provide arguments to support this contention: we give a reconstruction of the discovery of the incompleteness theorems, following the historical available sources, and pointing out the steps in which, we believe, Gödel's philosophical convictions were crucial. In section 1 we expose Gödel's Platonism as expressed in [Gödel 1947] and in [Gödel 1944]. The last section is included for conclusions and also to highlight Gödel's attempts to use his mathematical results to argue philosophically.

1 On Gödel's Platonism.

It has been said that Mathematicians are *Platonists* during working days, and some sort of *Formalists* during the weekend. Gödel was a Platonist even during the weekend.

A Platonistic philosophical standpoint accepts the existence of an ideal reality (in Plato's sense), an ideal reality whose existence is beyond space, time, and sense experience.

During the 1920's, the main philosophical views regarding the foundations of mathematics were, among others, Formalism (David Hilbert), Logical Positivism (Vienna Circle), and Intuitionism (L.E.J. Brouwer). None of these "schools" would accept the existence (at least in a wide open sense) of mathematical objects beyond formalisms and beyond human reasoning.

There Gödel was, a cautious Platonist, among them all, living in Vienna, attending discussions at the Café Reichstrat, well-informed about Hilbert's and Brouwer's programs.

Gödel's 1944 and 1947 essays are the only two philosophical essays (regarding mathematics) that he published in his lifetime. The first, *Russell's Mathematical Logic*, is dedicated to the study of Russell's logicism. The second, *What is Cantor's Continuum Problem?* (which was written after he proved the consistency of the General Continuum Hypothesis[1] with the axioms of Zermelo-Fraenkel[2] and the axiom of Choice), deals with the foundations of set theory.

In both papers, Gödel claims the necessity of new axioms "based on a hitherto unknown idea... for deciding certain questions of abstract set theory and even for certain related questions for the theory of real numbers."[3] This necessity finds its origin in, for example, the fact that for Gödel, even if the GCH turns out to be independent of the axioms of ZF (i.e., if the consistency of the negation of GCH, with ZF, is proved)[4], the question about the truth of GCH is meaningful.

Gödel supports this view from the mathematical and epistemological point of view. But in essence, for him "the set-theorical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. Hence its undecidability from

[1]GCH in what follows.
[2]ZF in what follows.
[3]See Gödel [1944].
[4]In 1963, P. Cohen proved this independence result.

the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality."[5]

Another declaration of Platonism is found in the last two pages of [Gödel 1964] (the second edition of [Gödel 1947]). Here, Gödel talks about *mathematical intuition.* He describes it as, "something as a perception [that we do have]... of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true." He goes on to say, "I don't see any reason why we should have less confidence in this kind of perception that in sense perception."

In [Gödel 1944], Gödel gives arguments against the mathematical need of Russell's *vicious circle principle.* For him, the vicious circle principle applies only if the mathematical entities involved (in the circular argument) are constructed by ourselves, but not "if, however , it is a question of objects that exist independently of our constructions..."

2 Reconstruction.

On Sunday September 7 of 1930 in Königsberg, Gödel announced in public,[6] for the first time, the existence of undecidable arithmetical propositions. Stating, "one can give examples of [arithmetical] propositions... that, while contentually true, are unprovable in the formal system of classical mathematics." [7] This announcement was made nearly a year after he presented his doctoral dissertation *On the completeness of the calculus of logic* at the University of Vienna.

After he obtained his Ph.D., at the beginning of 1930, Gödel started to look for an attractive problem that would serve as a topic for his *Habilitationsschrift.*[8] What could be a more attractive problem than one from Hilbert's 1900 list?

Gödel committed himself to the task of solving the second of these problems, namely, that of finding a finitary consistency proof (following Hilbert's Program) for the axioms of classical analysis. Surprisingly, it was when trying to solve this problem that he found the existence of undecidable propositions in arithmetic (under the assumption

[5] See [Göde 1947].

[6] It is recorded that Gödel made this announcement to Carnap 12 days before in Vienna.

[7] See [Dawson 97] p. 69.

[8] Paper that would serve as one of the requirements in order to obtain the academic degree of *Habilitation.*

of consistency). It may seem paradoxical that while he was looking for a partial solution of one of the main problems in Hilbert's Program, he found a result that brought down some of the principal aims of that program. We feel that this is not so surprising after all, considering Gödel's philosophical views as opposed to those of Hilbert and his followers (or even as opposed to the views of the logicists and the intuitionists). This difference led Gödel to the use of some additional metamathematical notions. To quote Gödel:

I may add that my objectivist conception of mathematics and metamathematics in general, ...was fundamental to my other work in logic [1931 incompleteness theorems]. *How indeed could one think of expressing metamathematics in the mathematical systems themselves, if the latter are to be considered to consist of meaningless symbols which acquire some substitute of meaning only through metamathematics ... it should be noted* [also] *that the heuristic principle of my construction of undecidable number theoretical propositions in the formal system of mathematics is the highly transfinite concept of* **objective mathematical truth** *as opposed to that of* **demonstrability**[9]

In what follows we give, as we understand, a reconstruction of the situation which led Gödel to his seminal incompleteness results of 1931.

Gödel started, as we have said, with the problem of giving a "finitary" consistency proof of classical analysis. According to Wang[10], Gödel claims to find this method of proof "mysterious". So aiming to divide difficulties he decided to split the problem in to two parts:

1. To give a consistency proof of classical analysis relative to the consistency of number theory.
2. To give a consistency proof of number theory in terms of the "finitist number theory".

The method for part 1 of the problem, would follow that of Hilbert used in *Grundlagen der Geometrie* [Hilbert 71], in which a relative consistency proof of Geometry with respect to Analysis is given by means of a formal interpretation.

Classical analysis can be viewed as the second order theory obtained from the second order Peano arithmetic by adding the axioms

[9] See [Feferman 1988] p.106
[10] See [Wang 1981].

of Extensionality and Comprehension. Comprehension stands for the axiom: for every second order formula on a single first order variable $\phi(x)$, the sentence

$$\exists C \forall x(x \in C \leftrightarrow \phi(x)),$$

and Extensionality for:

$$\forall A \forall B \forall x((x \in A \leftrightarrow x \in B) \to A = B)).$$

Where the variables A, B, C stand for second order variables.

Gödel tried to give an interpretation for this theory and then prove, from first order arithmetic, that in fact such an interpretation is a model for these axioms.

It seems natural to take as an interpretation the class of definable sets of natural numbers, i.e., to interpret each second order variable (real number) as a definable set of natural numbers. Gödel permitted himself even more: to interpret any set variable X, not by a definable set C_X, but by the set $\{\phi(m) : \phi(m)$ is a true sentence$\}$ (the set of true sentences of the form $\phi(m)$)[11], where $\phi(x)$ is a formula that defines C_X.

Gödel observed that in order to prove (from number theory) that this interpretation satisfies the Comprehension axioms, he needed to prove: for each formula $\Phi(x)$, there is a formula $\phi(x)$ such that for every natural number n,

$$\text{``}\phi(n) \text{ is a true sentence''} \leftrightarrow \Phi(n).$$

In order to carry out such a proof in first order arithmetic, one needs to express, in arithmetical terms, the sentence "$\phi(n)$ is a true sentence." That is to say, a definition of truth (in number theory) is needed.

It was Gödel's earlier idea to represent symbols by numbers[12] so that metamathematical relations would become mathematical ones. Therefore the problem of expressing the notion of arithmetical truth appeared feasible.

Based on his conversations with Gödel, Wang says that at this point, Gödel immediately ran into the Liar Paradox[13], in connection with the expressibility of the notion of truth. From here he visualized

[11] Actually, it is possible that Gödel had in mind to use the set of Gödel numbers of such sentences rather than the set of sentences itself. For our purpose either approach works.

[12] See [Wang 1981].

[13] *Idem*

the possible existence of undecidable propositions. We shall sketch Gödel's development in this direction.

By "arithmetical expressibility of truth" we mean: there is a first order formula $T(x)$ in the language for arithmetic such that for any arithmetical sentence σ we have,

$$T([\sigma])^{14} \iff \text{"}\sigma \text{ is a true sentence of arithmetic"}$$

In case such a formula $T(x)$ exists, Gödel realized that if by some diagonal method a sentence of the form $\neg T([\gamma])$ (where $[\gamma]$ is a code for the sentence $\neg T([\gamma])$ itself) can be constructed, one finds oneself with a particular form of the Liar Paradox: The sentence $\neg T([\gamma])$ (which, as we said, is equivalent to γ) will claim its own falsehood, as follows:

$$\begin{aligned}\neg T([\gamma]) &\iff \text{"}\gamma \text{ is a false sentence of arithmetic"}\\ &\iff \text{"}\neg T([\gamma]) \text{ is a false sentence of arithmetic"}\end{aligned}$$

So γ (or equivalently $\neg T([\gamma])$) is true if and only if it is false:

$$\begin{aligned}&\iff \text{"}\neg T([\gamma]) \text{ is a false sentence of arithmetic"}\\ &\iff \text{"}T([\gamma)] \text{ is a true sentence of arithmetic"}\\ &\iff \text{"}\gamma \text{ is a true sentence of arithmetic"}\\ &\iff \text{"}\neg T([\gamma]) \text{ is a true sentence of arithmetic"}\end{aligned}$$

But for Gödel, the Liar Paradox is nothing but a proof (by way of contradiction) that the notion of truth in a given language cannot be expressed in that language itself[15]. Here is the argument:

Let A be an individual that in time t makes the statement $Q :\equiv$ "Every statement that A makes in time t is false." Clearly this statement cannot be neither true nor false. Gödel says that Q is meaningless as stated, because the notion of truth must be relativized to a particular language. So A must specify a language (say L) and state $Q' :\equiv$ "Every statement of L that A makes, in time t, is a false statement in L." Now the "paradox" really becomes a proof (by way of contradiction) of the nonexpressibility, in the language L, of the notion "truth in L". For if this notion is expressible in L, then Q' would be expressible in L, which would lead to a contradiction:

$$Q' \text{ is false in } L \Leftrightarrow Q' \text{ is true in } L.$$

[14]By $[\sigma]$ is meant an arithmetical code that in some sense represents the sentence σ.
[15]See [Gödel 1934] section 7.

This general argument would serve as a proof of the nonexpressibility of the notion of arithmetical truth in arithmetic itself.

On the other hand, if the notion of "formal arithmetical proof" can be expressed in arithmetic, the existence of formally undecidable propositions will follow from the next argument:

Let α be the class of true arithmetical propositions, and β the class of formally provable ones.

If the notion of formal demonstrability is expressible in formal arithmetic, but that of arithmetical truth is not, then we must have $\alpha \neq \beta$. Now, by assuming the consistency as well as the correctness of number theory, we must also have $\beta \subset \alpha$. Therefore, there must be $\sigma \in \alpha \setminus \beta \neq \emptyset$, i.e., there must be an arithmetical true sentence that is not provable in formal arithmetic. And, since $\beta \subset \alpha$, its negation would also be formally unprovable.

Gödel was able to show that self-referential statements can be built in arithmetic by means of the arithmetization of the syntax and the use of primitive recursive functions. He also was able to show that the notion of "formal arithmetical proof" could be expressed in formal arithmetic. Therefore a "semantic proof" for the existence of formally undecidable propositions was found (just by following the argument above).

But Gödel went slightly further. He did not give the above semantic proof, but a syntactical proof not involving the ambiguous notion of truth that was so rejected by others during that time. The syntactical proof consists of exhibiting an undecidable sentence by constructing a self-referential (and true) sentence that would claim its own unprovability. A syntactical proof like that would definitively convince the skeptics.

3 From Philosophy to Mathematics.

In this final section we present arguments to support our understanding that Gödel's Platonistic philosophical standpoint, led him to the discovery of the incompleteness theorems.

In the previous section we outlined the path that Gödel took, and we also showed how this path led him to the point where he could clearly see the existence of undecidable propositions of number theory. Why did Gödel choose this path in order to give a consistency proof for classical analysis?

Our answer to this question is: Gödel chose this road due to his objective conception of mathematics and metamathematics, and due to his free use of transfinite reasoning. In order to support our answer, first observe that the heuristic principle involved in the construction of undecidable propositions is the highly transfinite concept of objective mathematical truth (as opposed to that of demonstrability) which caused great confusion before Gödel and Tarski[16]. Certainly, the construction given in the previous section is in need of the transfinite concept of truth in at least the following two occasions:

1. When considering as a model for interpretation the sets of the form $\{\phi(n) : \phi(n)$ is a true sentence of arithmetic $\}$.
2. When asking himself about the possible expressibility of the notion of truth.

In order to conceive a notion of truth as opposed to that of demonstrability, it seems that one is required to believe in the existence of objective mathematical concepts: a belief in which the question of the truth of an arithmetical sentence cannot be reduced to that of finding a formal proof, but to that of finding out if what the proposition states about numbers holds in the realm of numbers.

To summarize:

– Gödel's philosophical view assumes the independent existence of mathematical objects.
– This assumption makes him believe in a difference between the notions of demonstrability and truth.
– The use of the notion of truth as opposed to that of demonstrability is the heuristic principle in his construction of undecidable arithmetical propositions.

We would like to finish by pointing out that Gödel attempted to use his incompleteness results to support his Platonistic understanding of the mathematical world as opposed to other philosophical standpoints, especially the one expressed by the members of the Vienna circle (R. Carnap, M. Shlick, et al), and also the one expressed by David Hilbert. These philosophical views fall in what Gödel calls the "syntactical conception".

[16]For a complete study of the notion of truth in formal languages, see [Tarski 1983].

After Gödel's death, two of his unfinished articles were found. One called *Is Mathematics Syntax of Language?*[17](six versions) and the other *Some Basic Theorems on the Foundations of Mathematics and Their Implications*[18]. In these papers Gödel attempts to refute the following theses:

- Mathematical intuition can be replaced by conventions about the use of symbols and their application.
- In contradistinction to other sciences, which describe certain objects and facts, there do not exist any mathematical objects or facts. Mathematicals propositions are void of content.[19]

In [Gödel 1951] he also supports the claim that the human mind cannot be reduced to just a brain and its neuronal connections (if the consistency of the human mind is to be assumed).

Space does not permit a thorough discussion of these papers. But we outline an argument that captures the flavor of Gödel's ideas:

If the human mind can be reduced to a machine (brain and its neuronal connections), incompleteness may be applied (assuming the consistency of the machine), and so the existence of undecidable arithmetical proposition for the human being follows. But Gödel was the kind of person who belived that there are no limits for the human mind, i.e., there are no absolute unknown mathematical statements. In this he may agree with Hilbert's declaration:

For the mathematitian there is no Ignoramibus...
our credo avers:
We must know,
We shall know.

[17]See[Gödel 1953?].
[18]See [G"odel 1951].
[19]See [Gödel 1953?].

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